4-4 Transformations with Matrices

Main Ideas

- Use matrices to determine the coordinates of a translated or dilated figure.
- Use matrix multiplication to find the coordinates of a reflected or rotated figure.

New Vocabulary

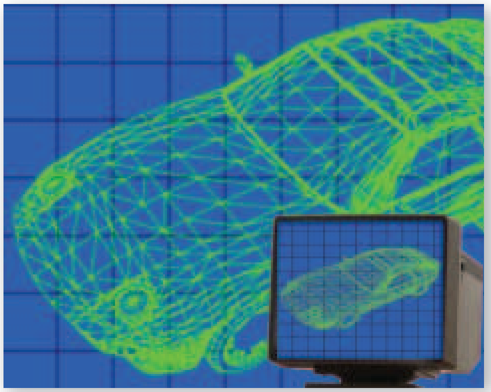
vertex matrix
transformation
preimage
image
translation
dilation
reflection
rotation

Reading Math

Coordinate Matrix
A matrix containing coordinates of a geometric figure is also called a *coordinate matrix.*

GET READY for the Lesson

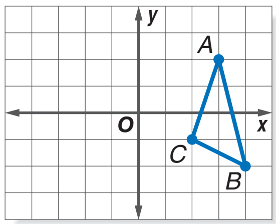
Computer animation creates the illusion of motion by using a succession of computer-generated still images. Computer animation is used to create movie special effects and to simulate images that would be impossible to show otherwise.



Complex geometric figures can be broken into simple triangles and then moved to other parts of the screen using matrices.

Translations and Dilations Points on a coordinate plane can be represented by matrices. The ordered pair (x, y) can be represented by the column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$. Likewise, polygons can be represented by placing all of the column matrices of the coordinates of the vertices into one matrix, called a **vertex matrix**.

Triangle ABC with vertices $A(3, 2)$, $B(4, -2)$, and $C(2, -1)$ can be represented by the following vertex matrix.



$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \triangle ABC = & \begin{bmatrix} 3 & 4 & 2 \\ 2 & -2 & -1 \end{bmatrix} \begin{array}{l} \leftarrow \text{x-coordinates} \\ \leftarrow \text{y-coordinates} \end{array} \end{array}$$

Notice that the triangle has 3 vertices and the vertex matrix has 3 columns. In general, the vertex matrix for a polygon with n vertices will have dimensions of $2 \times n$.

Matrices can be used to perform transformations. **Transformations** are functions that map points of a **preimage** onto its **image.**

One type of transformation is a translation. A **translation** occurs when a figure is moved from one location to another without changing its size, shape, or orientation. You can use matrix addition and a *translation matrix* to find the coordinates of a translated figure. The dimensions of a translation matrix should be the same as the dimensions of the vertex matrix.

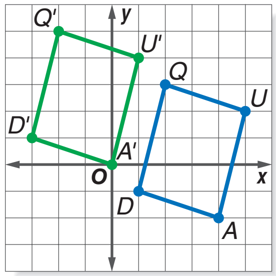
EXAMPLE Translate a Figure

1 **Find the coordinates of the vertices of the image of quadrilateral *QUAD* with *Q*(2, 3), *U*(5, 2), *A*(4, −2), and *D*(1, −1) if it is moved 4 units to the left and 2 units up. Then graph *QUAD* and its image *Q′U′A′D′*.**

Write the vertex matrix for quadrilateral *QUAD*. $\begin{bmatrix} 2 & 5 & 4 & 1 \\ 3 & 2 & -2 & -1 \end{bmatrix}$

To translate the quadrilateral 4 units to the left, add −4 to each x-coordinate. To translate the figure 2 units up, add 2 to each y-coordinate. This can be done by adding the translation matrix $\begin{bmatrix} -4 & -4 & -4 & -4 \\ 2 & 2 & 2 & 2 \end{bmatrix}$ to the vertex matrix of *QUAD*.

$$\underset{\text{Vertex Matrix of } QUAD}{\begin{bmatrix} 2 & 5 & 4 & 1 \\ 3 & 2 & -2 & -1 \end{bmatrix}} + \underset{\text{Translation Matrix}}{\begin{bmatrix} -4 & -4 & -4 & -4 \\ 2 & 2 & 2 & 2 \end{bmatrix}} = \underset{\text{Vertex Matrix of } Q'U'A'D'}{\begin{bmatrix} -2 & 1 & 0 & -3 \\ 5 & 4 & 0 & 1 \end{bmatrix}}$$



The vertices of *Q′U′A′D′* are *Q′*(−2, 5), *U′*(1, 4), *A′*(0, 0), and *D′*(−3, 1). *QUAD* and *Q′U′A′D′* have the same size and shape.

CHECK Your Progress

1. Find the coordinates of the vertices of the image of triangle *RST* with *R*(−1, 5), *S*(2, 1), and *T*(−3, 2) if it is moved 3 units to the right and 4 units up. Then graph *RST* and its image *R′S′T′*.

STANDARDIZED TEST EXAMPLE Find a Translation Matrix

2 Rectangle *A′B′C′D′* is the result of a translation of rectangle *ABCD*. A table of the vertices of each rectangle is shown. Find the coordinates of *D′*.

Rectangle *ABCD*	Rectangle *A′B′C′D′*
A(−4, 5)	*A′*(−1, 1)
B(1, 5)	*B′*(4, 1)
C(1, −2)	*C′*(4, −6)
D(−4, −2)	*D′*

A (−7, 2) **B** (−7, −6) **C** (−1, −6) **D** (−1, 2)

Test-Taking Tip

Sometimes you need to solve for unknown value(s) before you can solve for the value(s) requested in the question.

Read the Test Item

You are given the coordinates of the preimage and image of points *A*, *B*, and *C*. Use this information to find the translation matrix. Then you can use the translation matrix to find the coordinates of *D*.

Solve the Test Item

Step 1 Write a matrix equation. Let (c, d) represent the coordinates of *D*.

$$\begin{bmatrix} -4 & 1 & 1 & -4 \\ 5 & 5 & -2 & -2 \end{bmatrix} + \begin{bmatrix} x & x & x & x \\ y & y & y & y \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 & c \\ 1 & 1 & -6 & d \end{bmatrix}$$

$$\begin{bmatrix} -4+x & 1+x & 1+x & -4+x \\ 5+y & 5+y & -2+y & -2+y \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 & c \\ 1 & 1 & -6 & d \end{bmatrix}$$

Extra Examples at algebra2.com

Step 2 The matrices are equal, so corresponding elements are equal.

$-4 + x = -1$ Solve for x. $\qquad 5 + y = 1$ Solve for y.

$x = 3 \qquad y = -4$

Step 3 Use the values for x and y to find the values for $D'(c, d)$.

$-4 + 3 = c \qquad -2 + (-4) = d$

$-1 = c \qquad -6 = d$

So the coordinates for D are $(-1, -6)$, and the answer is C.

CHECK Your Progress

Triangle XYZ	Triangle $X'Y'Z'$
$X(3, -1)$	$X'(1, 0)$
$Y(-4, 2)$	$Y'(-6, 3)$
$Z(5, 1)$	Z'

2. Triangle $X'Y'Z'$ is the result of a translation of triangle XYZ. Find the coordinates of Z' using the information shown in the table.

F (3, 2) **G** (7, 2) **H** (7, 0) **J** (3, 0)

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Study Tip

Dilations

In a dilation, all linear measures of the image change in the same ratio. The image is similar to the preimage.

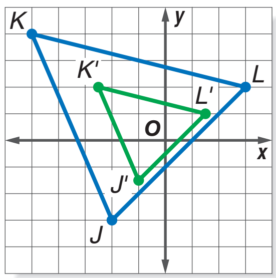
When a figure is enlarged or reduced, the transformation is called a **dilation**. A dilation is performed relative to its center. Unless otherwise specified, the center is the origin. You can use scalar multiplication to perform dilations.

EXAMPLE 3 Dilation

Dilate $\triangle JKL$ with $J(-2, -3)$, $K(-5, 4)$, and $L(3, 2)$ so that its perimeter is half the original perimeter. Find the coordinates of the vertices of $\triangle J'K'L'$.

If the perimeter of a figure is half the original perimeter, then the lengths of the sides of the figure will be one-half the measure of the original lengths. Multiply the vertex matrix by the scale factor of $\frac{1}{2}$.

$$\frac{1}{2}\begin{bmatrix} -2 & -5 & 3 \\ -3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & 2 & 1 \end{bmatrix}$$



The coordinates of the vertices of $\triangle J'K'L'$ are $J'\left(-1, -\frac{3}{2}\right)$, $K'\left(-\frac{5}{2}, 2\right)$, and $L'\left(\frac{3}{2}, 1\right)$.

CHECK Your Progress

3. Dilate rectangle $MNPQ$ with $M(4, 4)$, $N(4, 12)$, $P(8, 4)$, and $Q(8, 12)$ so that its perimeter is one fourth the original perimeter. Find the coordinates of the vertices of rectangle $M'N'P'Q'$.

Reflections and Rotations A **reflection** maps every point of a figure to an image across a line of symmetry using a *reflection matrix*.

CONCEPT SUMMARY *Reflection Matrices*

For a reflection over the:	x-axis	y-axis	line $y = x$
Multiply the vertex matrix on the left by:	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

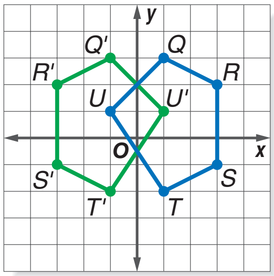
EXAMPLE 4 Reflection

Find the coordinates of the vertices of the image of pentagon *QRSTU* with *Q*(1, 3), *R*(3, 2), *S*(3, −1), *T*(1, −2), and *U*(−1, 1) after a reflection across the *y*-axis.

Write the ordered pairs as a vertex matrix. Then multiply the vertex matrix by the reflection matrix for the *y*-axis.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 3 & 1 & -1 \\ 3 & 2 & -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -3 & -3 & -1 & 1 \\ 3 & 2 & -1 & -2 & 1 \end{bmatrix}$$

Notice that the preimage and image are congruent. Both figures have the same size and shape.



CHECK Your Progress

4. Find the coordinates of the vertices of the image of pentagon *QRSTU* after a reflection across the *x*-axis.

A **rotation** occurs when a figure is moved around a center point, usually the origin. To determine the vertices of a figure's image by rotation, multiply its vertex matrix by a *rotation matrix*.

CONCEPT SUMMARY — *Rotation Matrices*

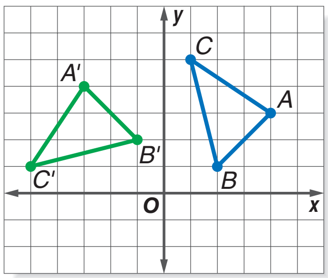
For a counterclockwise rotation about the origin of:	90°	180°	270°
Multiply the vertex matrix on the left by:	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

EXAMPLE 5 Rotation

Find the coordinates of the vertices of the image △*ABC* with *A*(4, 3), *B*(2, 1), and *C*(1, 5) after it is rotated 90° counterclockwise about the origin.

Write the ordered pairs in a vertex matrix. Then mutiply the vertex matrix by the rotation matrix.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 & 1 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} -3 & -1 & -5 \\ 4 & 2 & 1 \end{bmatrix}$$



The coordinates of the vertices of △*A′B′C′* are *A′*(−3, 4), *B′*(−1, 2), and *C′*(−5, 1). The image is congruent to the preimage.

CHECK Your Progress

5. Find the coordinates of the vertices of the image of △*XYZ* with *X*(−5, −6), *Y*(−1, −3), and *Z*(−2, −4) after it is rotated 180° counterclockwise about the origin.

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✓ CHECK Your Understanding

Example 1 (pp. 185–186)

Triangle ABC with vertices $A(1, 4)$, $B(2, -5)$, and $C(-6, -6)$ is translated 3 units right and 1 unit down.

1. Write the translation matrix.

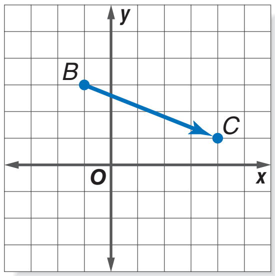
2. Find the coordinates of $\triangle A'B'C'$.

3. Graph the preimage and the image.

Example 2 (pp. 186–187)

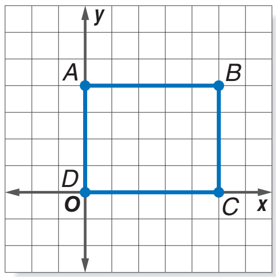
4. STANDARDIZED TEST PRACTICE A point is translated from B to C as shown at the right. If a point at $(-4, 3)$ is translated in the same way, what will be its new coordinates?

A $(3, 4)$ **B** $(1, 1)$ **C** $(-8, 8)$ **D** $(1, 6)$



Example 3 (p. 187)

For Exercises 5–11, use the rectangle at the right.



5. Write the coordinates in a vertex matrix.

6. Find the coordinates of the image after a dilation by a scale factor of 3.

7. Find the coordinates of the image after a dilation by a scale factor of $\frac{1}{2}$.

Example 4 (p. 188)

8. Find the coordinates of the image after a reflection over the x-axis.

9. Find the coordinates of the image after a reflection over the y-axis.

10. Find the coordinates of the image after a rotation of 180°.

Example 5 (p. 188)

11. Find the coordinates of the image after a rotation of 270°.

Exercises

HOMEWORK HELP

For Exercises	See Examples
12, 13	1
14, 15	2
16, 17	3
18, 19	4
20, 21	5

Write the translation matrix for each figure. Then find the coordinates of the image after the translation. Graph the preimage and the image on a coordinate plane.

12. $\triangle DEF$ with $D(1, 4)$, $E(2, -5)$, and $F(-6, -6)$, translated 4 units left and 2 units up

13. $\triangle MNO$ with $M(-7, 6)$, $N(1, 7)$, and $O(-3, 1)$, translated 2 units right and 6 units down

14. Rectangle $RSUT$ with vertices $R(-3, 2)$, $S(1, 2)$, $U(1, -1)$, $T(-3, -1)$ is translated so that T' is at $(-4, 1)$. Find the coordinates of R' and U'.

15. Triangle DEF with vertices $D(-2, 2)$, $E(3, 5)$, and $F(5, -2)$ is translated so that D' is at $(1, -5)$. Find the coordinates of E' and F'.

Write the vertex matrix for each figure. Then find the coordinates of the image after the dilation. Graph the preimage and the image on a coordinate plane.

16. $\triangle ABC$ with $A(0, 2)$, $B(1.5, -1.5)$, and $C(-2.5, 0)$ is dilated so that its perimeter is three times the original perimeter.

17. $\triangle XYZ$ with $X(-6, 2)$, $Y(4, 8)$, and $Z(2, -6)$ is dilated so that its perimeter is one half times the original perimeter.

Write the vertex matrix and the reflection matrix for each figure. Then find the coordinates of the image after the reflection. Graph the preimage and the image on a coordinate plane.

18. The vertices of $\triangle XYZ$ are $X(1, -1)$, $Y(2, -4)$, and $Z(7, -1)$. The triangle is reflected over the line $y = x$.

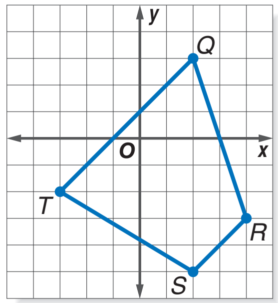
19. The vertices of rectangle $ABDC$ are $A(-3, 5)$, $B(5, 5)$, $D(5, -1)$, and $C(-3, -1)$. The rectangle is reflected over the x-axis.

Write the vertex matrix and the rotation matrix for each figure. Then find the coordinates of the image after the rotation. Graph the preimage and the image on a coordinate plane.

20. Parallelogram $DEFG$ with $D(2, 4)$, $E(5, 4)$, $F(4, 1)$, and $G(1, 1)$ is rotated 270° counterclockwise about the origin.

21. $\triangle MNO$ with $M(-2, -6)$, $N(1, 4)$, and $O(3, -4)$ is rotated 180° counterclockwise about the origin.

For Exercises 22–24, refer to the quadrilateral *QRST* shown at the right.



22. Write the vertex matrix. Multiply the vertex matrix by -1.

23. Graph the preimage and image.

24. What type of transformation does the graph represent?

25. A triangle is rotated 90° counterclockwise about the origin. The coordinates of the vertices are $J'(-3, -5)$, $K'(-2, 7)$, and $L'(1, 4)$. What were the coordinates of the triangle in its original position?

26. A triangle is rotated 90° clockwise about the origin. The coordinates of the vertices are $F'(2, -3)$, $G'(-1, -2)$, and $H'(3, -2)$. What were the coordinates of the triangle in its original position?

27. A quadrilateral is reflected across the y-axis. The coordinates of the vertices are $P'(-2, 2)$, $Q'(4, 1)$, $R'(-1, -5)$, and $S'(-3, -4)$. What were the coordinates of the quadrilateral in its original position?

Real-World Link

Douglas Engelbart invented the "X-Y position indicator for a display system" in 1964. He nicknamed this invention "the mouse" because a tail came out the end.

Source: about.com

For Exercises 28–31, use rectangle *ABCD* with vertices $A(-4, 4)$, $B(4, 4)$, $C(4, -4)$, and $D(-4, -4)$.

28. Find the coordinates of the image in matrix form after a reflection over the x-axis followed by a reflection over the y-axis.

29. Find the coordinates of the image in matrix form after a 180° rotation about the origin.

30. Find the coordinates of the image in matrix form after a reflection over the line $y = x$.

31. What do you observe about these three matrices? Explain.

TECHNOLOGY For Exercises 32 and 33, use the following information.
As you move the mouse for your computer, a corresponding arrow is translated on the screen. Suppose the position of the cursor on the screen is given in inches with the origin at the bottom left-hand corner of the screen.

32. Write a translation matrix that can be used to move the cursor 3 inches to the right and 4 inches up.

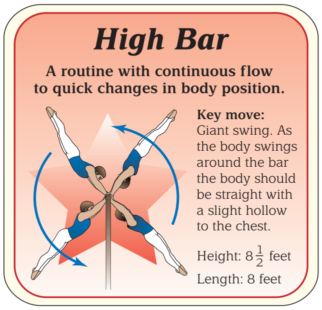
33. If the cursor is currently at (3.5, 2.25), what are the coordinates of the position after the translation?

LANDSCAPING **For Exercises 34 and 35, use the following information.** A garden design is plotted on a coordinate grid. The original plan shows a fountain with vertices at $(-2, -2)$, $(-6, -2)$, $(-8, -5)$, and $(-4, -5)$. Changes to the plan now require that the fountain's perimeter be three-fourths that of the original.

34. Determine the coordinates for the vertices of the fountain.

35. The center of the fountain was at $(-5, -3.5)$. What will be the coordinates of the center after the changes in the plan have been made?

36. **GYMNASTICS** The drawing at the right shows four positions of a man performing the giant swing in the high bar event. Suppose this drawing is placed on a coordinate grid with the hand grips at $H(0, 0)$ and the toe of the figure in the upper right corner at $T(7, 8)$. Find the coordinates of the toes of the other three figures, if each successive figure has been rotated 90° counterclockwise about the origin.



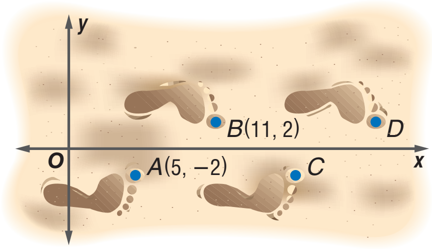
FOOTPRINTS **For Exercises 37–40, use the following information.** The combination of a reflection and a translation is called a *glide reflection*. An example is a set of footprints.

37. Describe the reflection and transformation combination shown at the right.

38. Write two matrix operations that can be used to find the coordinates of point C.

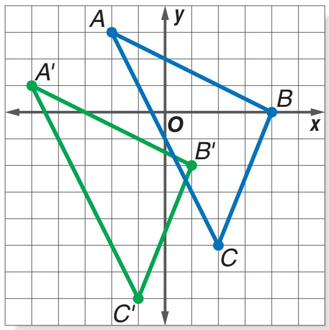
39. Does it matter which operation you do first? Explain.

40. What are the coordinates of the next two footprints?



EXTRA PRACTICE
See pages 898, 929.
Math Online
Self-Check Quiz at
algebra2.com

41. Write the translation matrix for $\triangle ABC$ and its image $\triangle A'B'C'$ shown at the right.



42. Compare and contrast the size and shape of the preimage and image for each type of transformation. For which types of transformations are the images congruent to the preimage?

H.O.T. Problems

43. **OPEN ENDED** Write a translation matrix that moves $\triangle DEF$ up and left.

44. **CHALLENGE** Do you think a matrix exists that would represent a reflection over the line $x = 3$? If so, make a conjecture and verify it.

45. **REASONING** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

The image of a dilation is congruent to its preimage.

46. ***Writing in Math*** Use the information about computer animation on page 185 to explain how matrices can be used with transformations in computer animation. Include an example of how a figure with 5 points (coordinates) changes as a result of repeated dilations.

STANDARDIZED TEST PRACTICE

47. ACT/SAT Triangle ABC has vertices with coordinates $A(-4, 2)$, $B(-4, -3)$, and $C(3, -2)$. After a dilation, triangle $A'B'C'$ has coordinates $A'(-12, 6)$, $B'(-12, -9)$, and $C'(9, -6)$. How many times as great is the perimeter of $\triangle A'B'C'$ as that of $\triangle ABC$?

A 3

B 6

C 12

D $\frac{1}{3}$

48. REVIEW Melanie wanted to find 5 consecutive whole numbers that add up to 95. She wrote the equation $(n - 2) + (n - 1) + n + (n + 1) + (n + 2) = 98$. What does the variable n represent in the equation?

F The least of the 5 whole numbers

G The middle of the 5 whole numbers

H The greatest of the 5 whole numbers

J The difference between the least and the greatest of the 5 whole numbers.

Spiral Review

Determine whether each matrix product is defined. If so, state the dimensions of the product. (Lesson 4-3)

49. $A_{2 \times 3} \cdot B_{3 \times 2}$

50. $A_{4 \times 1} \cdot B_{2 \times 1}$

51. $A_{2 \times 5} \cdot B_{5 \times 5}$

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*. (Lesson 4-2)

52. $2\begin{bmatrix} 4 & 9 & -8 \\ 6 & -11 & -2 \\ 12 & -10 & 3 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

53. $4\begin{bmatrix} 3 & 4 & -7 \\ 6 & -9 & -2 \\ -3 & 1 & 3 \end{bmatrix} - \begin{bmatrix} -8 & 6 & -4 \\ -7 & 10 & 1 \\ -2 & 1 & 5 \end{bmatrix}$

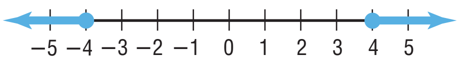
Graph each relation or equation and find the domain and range. Then determine whether the relation or equation is a function. (Lesson 2-1)

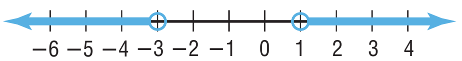
54. $(3, 5), (4, 6), (5, -4)$

55. $x = -5y + 2$

56. $x = y^2$

Write an absolute value inequality for each graph. (Lesson 1-6)

57. 

58. 

59. BUSINESS Reliable Rentals rents cars for \$12.95 per day plus 15¢ per mile. Luis Romero works for a company that limits expenses for car rentals to \$90 per day. How many miles can Mr. Romero drive each day? (Lesson 1-5)

GET READY for the Next Lesson

PREREQUISITE SKILL **Use cross products to solve each proportion.**

60. $\frac{x}{8} = \frac{3}{4}$

61. $\frac{4}{20} = \frac{1}{m}$

62. $\frac{2}{3} = \frac{a}{42}$

63. $\frac{2}{y} = \frac{8}{9}$

64. $\frac{4}{n} = \frac{6}{2n - 3}$

65. $\frac{x}{5} = \frac{x + 1}{8}$